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Hamiltonian formulation of noncommutative D3-brane

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Abstract: Lagrangians of the abelian Gauge Theory and its dual are related in terms of a shifted action. We show that in $d = 4$ constrained hamiltonian formulation of the shifted action yields hamiltonian description of the dual theory, without referring to its lagrangian. We apply this method, at the first order in the noncommutativity parameter θ, to the noncommutative U(1) gauge theory possessing spatial noncommutativity. Its dual theory is effectively a space-time noncommutative U(1) gauge theory. However, we obtain a hamiltonian formulation where time is commuting. Space-time noncommutative D3-brane worldvolume hamiltonian is derived as the dual of space noncommutative U(1) gauge theory. We show that a BPS like bound can be obtained and it is saturated for configurations which are the same with the ordinary D3-brane BIon and dyon solutions.

Keywords: D-branes, String Duality, Non-Commutative Geometry.

Contents

1. Introduction

Noncommutative and ordinary gauge theory descriptions of D-branes in a constant background B-field are equivalent perturbatively in the noncommutativity parameter $\theta^{\mu\nu}$ [1]. To find evidence that this equivalence is valid even nonperturbatively, noncommutative D3-brane BIon and dyon are studied in [2]. Noncommutative D3-brane BIon configuration is attained when open string metric satisfies $G_{MN} = \mathrm{diag}(-1, 1, \ldots, 1)$ where $M, N = 0, 1, \ldots, 9$. This geometry is accomplished allowing a background B-field on D3-brane worldvolume, producing a noncommutativity parameter $\theta^{01} \neq 0$ and $\theta^{02} = \theta^{03} = \theta^{ij} = 0$, where $i, j = 1, 2, 3$. At the lowest order in the string slope parameter α' and for slowly varying fields noncommutative D3-brane is described in terms of noncommutative U(1) gauge theory. Non-vanishing θ^{01} leads to noncommutative time in terms of Moyal bracket. Thus the ordinary hamiltonian formalism of this system is obscure. However, owing to the fact that the theory is invariant under translations, an energy density is derived from lagrangian which is utilized to write a Bogomol'nyi-Prasad-Sommerfeld (BPS) like bound.

When time is noncommuting with the spatial coordinates the usual hamiltonian methods are not applicable. To overcome this difficulty in [3] a new method is developed introducing a spurious time like variable. In this approach the energy is the same as the one derived from lagrangian path integral formalism of the original theory.

We would like to examine if one can find a hamiltonian in an ordinary phase space for space-time noncommutative D3-brane. It would yield a well defined energy. This is possible if the lagrangian with noncommutative time can be considered as an object derived from an original theory whose time variable is commuting. Indeed, in string theories noncommutative time parameter usually appears in the actions which are (S) dual of initial theories with commuting time [4]. Similarly, in [5] noncomutative U(1) gauge theory with the noncommutativity parameter $\tilde{\theta}^{ij} = 0$, $\tilde{\theta}^{0i} \neq 0$, is established as the dual theory of the one whose noncommutativity parameter satisfies $\theta^{ij} \neq 0$, $\theta^{0i} = 0$.

Legendre transforming the abelian gauge theory lagrangian in terms of dual gauge field and performing path integration of the shifted action over the field strength lead to the lagrangian formalism of the dual theory. We will show that constrained hamiltonian structure [7] of the shifted abelian gauge theory action in $d = 4$ provides us hamiltonian formalism of the dual theory without referring to its lagrangian. This method of bypassing lagrangian of the dual theory to derive its phase space formalism is interesting in itself. Moreover, it may be very useful to treat the theories whose lagrangian formalism is given in terms of (effectively) noncommuting time variable. In fact, we apply it to noncommutative U(1) gauge theory with spatial noncommutativity and obtain phase space formulation of dual gauge theory whose time coordinate is noncommuting in terms of Moyal bracket. On the other hand the dual lagrangian is known and it is originated from a theory with a commuting time variable. Thus, one can treat time variable as commuting, although effectively time and spatial coordinates satisfy a non-vanishing Moyal bracket. Hamiltonian and phase space constraints of the both approaches coincide. We deal only with the first order approximation in noncommutativity parameter $\theta^{\mu\nu}$.

Once hamiltonian formulation of the dual theory is achieved, noncommutative D3-brane worldvolume hamiltonian formalism in static gauge can be written directly. We define θ dependent (noncommutative) fields in terms of the usual phase space fields and obtain a BPS like bound on energy. Saturation of this bound leads to equations in terms of θ dependent fields. These are solved when commutative fields satisfy ordinary BPS equations. This is similar to the observation that linearized and full Dirac-Born-Infeld (DBI) theories possess the same BPS states [6] with the same energy. However, in our case energies differ.

In section 2 we study abelian Gauge Theory in $d = 4$. We show that the appropriately shifted lagrangian can be studied as a constrained hamiltonian system and reduced phase space method leads to hamiltonian formulation of the dual theory. Thus, without referring to the lagrangian we can obtain the hamiltonian formalism of the dual theory.

In section 3 the method illustrated in section 2 is utilized to derive hamiltonian formulation of the noncommutative U(1) gauge theory whose time coordinate is noncommuting with the spatial ones in terms of Moyal bracket. We treat the gauge theory with spatial noncommutativity as the original one. Considering time as commuting in the dual theory a phase space formulation is found in terms of the usual methods. These two hamiltonian descriptions coincide.

In section 4 we derive the hamiltonian for D3-brane worldvolume with a scalar field. We introduce θ dependent (noncommutative) fields to put this hamiltonian in a form suitable to derive a BPS like bound on energy. Conditions to saturate this bound are discussed. It is shown that these conditions are solved when commuting fields are taken as D3-brane BIon or dyon solutions. For some specific choices of fields, charges taking part in BPS bound are shown to be topological, although θ dependent.

The results obtained are discussed in the last section.

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2. Abelian gauge theory

Abelian gauge theory action in $d = 4$ with the Minkowski metric $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, is

$$S_o = -\frac{1}{4g^2}\int d^4x F_{\mu\nu}F^{\mu\nu}\,, \tag{2.1}$$

where $F = dA$. To implement duality we introduce the dual gauge field A_D and deal with the shifted action

$$S_m = \int d^4x \left(\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\mu A_D^\nu F^{\rho\sigma}\right). \tag{2.2}$$

Here we treat F as an independent variable without requiring any relation with the gauge field A. Performing path integral over F, which is equivalent to solve the equations of motion for F in terms of A_D and replace it in the action, leads to the dual action

$$S_D = -\frac{g^2}{4}\int d^4x F_D{}^{\mu\nu}F_{D\mu\nu} \tag{2.3}$$

where $F_D = dA_D$. Constraint hamiltonian structures resulting from S_o and S_D are related by canonical transformations [8].

We would like to study canonical formulation of S_m to demonstrate that hamiltonian and constraints of the dual action S_D result directly. Definition of canonical momenta

$$P_{\mu\nu} = \frac{\delta S_m}{\delta(\partial_0 F^{\mu\nu})}; \qquad P_{D\mu} = \frac{\delta S_m}{\delta(\partial_0 A_D^\mu)}\,, \tag{2.4}$$

leads to the primary constraints,

$$P_{\mu\nu} \approx 0\,, \tag{2.5}$$

$$P_{D0} \approx 0\,, \tag{2.6}$$

$$\phi_i \equiv P_{Di} + \frac{1}{2}\epsilon_{ijk}F^{jk} \approx 0\,, \tag{2.7}$$

where $i, j, k = 1, 2, 3$, and "$\approx$" denotes that constraints are weakly vanishing. The related canonical hamiltonian is

$$H_{mc} = \int d^3x\ \left[\frac{1}{2g^2}F^{0i}F_{0i} + \frac{1}{4g^2}F^{ij}F_{ij} - \frac{1}{2}\epsilon_{ijk}\partial^i A_D{}^0 F^{jk} + \epsilon_{ijk}\partial^i A_D{}^j F^{0k}\right]. \tag{2.8}$$

By adding the primary constraints (2.5)–(2.7) to the canonical hamiltonian, in terms of some Lagrange multipliers α_i, β, λ_{ij}, κ_i, one obtains the extended hamiltonian [7]

$$H_{me} = H_{mc} + \int d^3x\ \left[\alpha_i P_{0i} + \beta P_{D0} + \lambda_{ij}P_{ij} + \kappa_i\phi_i\right]. \tag{2.9}$$

Denote that we use the definition

$$\frac{\partial F_{\mu\nu}}{\partial F_{\rho\sigma}} = \delta_\mu{}^\rho\delta_\nu{}^\sigma - \delta_\mu{}^\sigma\delta_\nu{}^\rho\,,$$

to calculate Poisson brackets of the primary constraints (2.5)–(2.7) with H_{me}, which lead to the secondary constraints

$$\epsilon_{ijk}\partial^i F^{jk} \approx 0\,, \tag{2.10}$$

$$\frac{1}{g^2}F_{0i} + \epsilon_{ijk}\partial^j A_D{}^k \approx 0\,. \tag{2.11}$$

Constraints terminate here. The constraint (2.6) is first class and the rest (2.5), (2.7), (2.10), (2.11) are second class. In the reduced phase space, obtained by setting all the second class constraints equal to zero strongly and solving F, P in terms of F_D, P_D, the canonical hamiltonian (2.8) becomes

$$H_D = \int d^3x \left[\frac{1}{2g^2}P_{Di}P_{Di} + \frac{g^2}{4}F_{Dij}F_{Dij}\right]. \tag{2.12}$$

Moreover, there are the first class constraints

$$P_{D0} \approx 0\,, \qquad \partial_i P_{Di} \approx 0\,. \tag{2.13}$$

Obviously this is the same with the constrained hamiltonian formalism of the dual theory (2.3). Therefore we demonstrated that one can obtain constrained hamiltonian formulation of the dual theory beginning from the shifted action (2.2) bypassing the dual lagrangian (2.3).

3. Noncommutative U(1) gauge theory

Noncommuting variables should be treated as operators. However, one can retain them commuting under the usual product and introduce noncommutativity in terms of the star product

$$* \equiv \exp\frac{i\theta^{\mu\nu}}{2}\left(\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu - \overleftarrow{\partial}_\nu\overrightarrow{\partial}_\mu\right), \tag{3.1}$$

where $\theta^{\mu\nu}$ is a constant parameter. Now, the coordinates x^μ satisfy the Moyal bracket

$$x^\mu * x^\nu - x^\nu * x^\mu = \theta^{\mu\nu}\,. \tag{3.2}$$

Noncommutative U(1) gauge theory is given by the action

$$S_{nc} = -\frac{1}{4g^2}\int d^4x \widehat{F}_{\mu\nu} * \widehat{F}^{\mu\nu} \tag{3.3}$$

where we defined

$$\widehat{F}_{\mu\nu} = \partial_\mu\widehat{A}_\nu - \partial_\nu\widehat{A}_\mu - i\widehat{A}_\mu * \widehat{A}_\nu + i\widehat{A}_\nu * \widehat{A}_\mu\,. \tag{3.4}$$

Seiberg and Witten [1] showed that noncommutative gauge fields $\widehat{A}_\mu$, noncommutative gauge parameter $\widehat{\lambda}$ and the commuting ones A_μ, λ are related under the gauge transformations $\widehat{\delta}$, δ as

$$\widehat{A}(A) + \widehat{\delta}_{\widehat{\lambda}}\widehat{A}(A) = \widehat{A}(A + \delta_\lambda A)\,. \tag{3.5}$$

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At the first order in $\theta^{\mu\nu}$ it is solved to yield

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - \theta^{\rho\sigma} A_\rho \partial_\sigma F_{\mu\nu} \,. \tag{3.6}$$

Thus the action (3.3) can be written at the first order in $\theta^{\mu\nu}$ as

$$S_{nc} = -\frac{1}{4g^2} \int d^4x \left(F_{\mu\nu} F^{\mu\nu} + 2\theta^{\mu\nu} F_{\nu\rho} F^{\rho\sigma} F_{\sigma\mu} - \frac{1}{2} \theta^{\mu\nu} F_{\nu\mu} F_{\rho\sigma} F^{\sigma\rho} \right) . \tag{3.7}$$

To implement duality transformation deal with the shifted action [5]

$$S = -S_{nc} + \frac{1}{2} \int d^4x \ {A_D}^\mu \epsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} \tag{3.8}$$

where F and A_D are taken as independent field variables.

As in the commutative case dual action can be found by solving the field equations for F in terms of $F_D = dA_D$ and plugging it in the action (3.8) leading to

$$\tilde{S} = -\frac{g^2}{4} \int d^4x \left({F_D}^{\mu\nu} F_{D\mu\nu} + 2\tilde{\theta}^{\mu\nu} F_{D\nu\rho} F_D^{\rho\sigma} F_{D\sigma\mu} - \frac{1}{2} \tilde{\theta}^{\mu\nu} F_{D\nu\mu} F_{D\rho\sigma} F_D^{\sigma\rho} \right) \tag{3.9}$$

where $\tilde{\theta}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} \theta_{\rho\sigma}$. At the first order in $\tilde{\theta}$ it can be derived from the action

$$\tilde{S} = -\frac{g^2}{4} \int d^4x \widehat{F}_{D\mu\nu} \tilde{*} \widehat{F}_D^{\mu\nu} \,, \tag{3.10}$$

where $\tilde{*}$ is given by (3.1) by replacing θ with $\tilde{\theta}$. For $\theta^{0i} = 0$ and $\theta^{ij} \neq 0$ the dual theory is a gauge theory whose time variable is noncommuting in terms of the Moyal bracket (3.2) with $\tilde{*}$, because $\tilde{\theta}^{oi} \neq 0$, $\tilde{\theta}^{ij} = 0$. For a noncommuting time canonical formalism is obscure. Thus we would like to bypass the dual action (3.9) to obtain a phase space formulation of the dual theory using the method illustrated in section 2.

Let $\theta^{ij} \neq 0$ and $\theta^{i0} = 0$ in the action (3.8). Definition of canonical momenta

$$P_{\mu\nu} = \frac{\delta S}{\delta(\partial_0 F^{\mu\nu})} \tag{3.11}$$

$$P_{D\mu} = \frac{\delta S}{\delta(\partial_0 {A_D}^\mu)} \tag{3.12}$$

leads to the primary constraints

$$P_{\mu\nu} \approx 0 \,, \tag{3.13}$$

$$P_{D0} \approx 0 \,, \tag{3.14}$$

$$P_{Di} + \frac{1}{2} \epsilon_{ijk} F_{jk} \approx 0 \tag{3.15}$$

and the canonical hamiltonian

$$\begin{aligned} H_c = \int d^3x \Big[& -\frac{1}{2} \epsilon_{ijk} \partial^i {A_D}^0 F^{jk} + \epsilon_{ijk} \partial^i {A_D}^j F^{0k} + \frac{1}{2g^2} F_{0i} F^{0i} + \\ & + \frac{1}{4g^2} F_{ij} F^{ij} + \frac{1}{g^2} F^{0i} F_{ij} \theta^{jk} F_{k0} + \frac{1}{2g^2} F^{ij} F_{jk} \theta^{kl} F_{li} - \\ & - \frac{1}{4g^2} \theta^{ij} F_{ji} F_{0l} F^{l0} - \frac{1}{8g^2} \theta^{ij} F_{ji} F_{kl} F^{lk} \Big] . \end{aligned} \tag{3.16}$$

Preserving the primary constraints (3.13)–(3.15) in time leads to secondary constraints

$$\epsilon_{ijk}\partial^i F^{jk} \approx 0 \tag{3.17}$$

and

$$\Psi_i \equiv F^{0i} - F_{ij}\theta^{jk}F_{k0} - F^{0j}F_{jk}\theta^{ki} - \frac{1}{2}\theta^{jk}F_{kj}F_{0i} - g^2\epsilon_{ijk}\partial^j A_D{}^k \approx 0\,, \tag{3.18}$$

which do not yield new constraints. One can check that (3.14) is first class and the other constraints (3.13), (3.15), (3.17), (3.18) are second class. In the reduced phase space where second class constraints strongly vanish, the canonical hamiltonian (3.16) becomes

$$\begin{aligned} H_{nD} = \int d^3x \Big[&\frac{g^2}{4}F_{Dij}^2 + \frac{1}{2g^2}P_{Di}^2 - \frac{1}{2g^2}\epsilon_{ijk}\theta^{ij}P_D^kP_{Dl}^2 - \\ &- \frac{g^2}{4}\epsilon_{ijk}\theta^{ij}P_D^kF_{Dlm}^2 - g^2F_{Dij}P_D^j\theta^{ik}\epsilon_{klm}F_D^{lm}\Big]\,, \end{aligned} \tag{3.19}$$

if we solve F, P in terms of F_D and P_D. Moreover, there are still the constraints

$$\partial_i P_{Di} = 0\,, \qquad P_{D0} = 0\,, \tag{3.20}$$

which are first class. This hamiltonian can be written in terms of $\tilde\theta^{0i} = \epsilon^{ijk}\theta_{jk}$ as

$$\begin{aligned} H_{nD} = \int d^3x \Big[&\frac{g^2}{4}F_{Dij}^2 + \frac{1}{2g^2}P_{Di}^2 - \frac{1}{2g^2}\tilde\theta_{0i}P_D^iP_{Dj}^2 - \\ &- \frac{g^2}{4}\tilde\theta_{0i}P_D^iF_{Djk}^2 + g^2\tilde\theta^{0i}F_{Dji}F_{Djk}P_D^k\Big]\,. \end{aligned} \tag{3.21}$$

On the other hand, although the dual action (3.10) possesses a noncommuting time variable in terms of the Moyal bracket (3.2) given by $\tilde *$, it is originated from the action (3.7) whose time coordinate is commuting. We wonder what would be the phase space structure if we treat time coordinate as commuting in the action (3.9) written in components as

$$\begin{aligned} S_d = -g^2\int d^4x \Big[&\frac{1}{2}F_{D0i}F_{D0i} - \frac{1}{4}F_{Dij}F_{Dij} - \frac{1}{2}\tilde\theta^{0i}F_{Di0}F_{D0j}F_{D0j} - \tilde\theta^{0i}F_{Dij}F_{Djk}F_{Dk0} + \\ &+ \frac{1}{4}\tilde\theta^{0i}F_{Di0}F_{Djk}F_{Dkj}\Big]\,. \end{aligned} \tag{3.22}$$

Definition of the spatial components of momentum

$$\begin{aligned} P_{Di} = \frac{\delta\tilde S}{\delta(\partial_0 A_{Di})} = g^2\Big[&F_{D0i} - \frac{1}{2}\tilde\theta^{0i}F_{D0j}F_{D0j} + -\tilde\theta^{0j}F_{Dj0}F_{D0i} + \tilde\theta^{0k}F_{Dkj}F_{Dji} \\ &+ \frac{1}{4}\tilde\theta^{0i}F_{Djk}F_{Dkj}\Big] \end{aligned} \tag{3.23}$$

can be solved to find $\partial_0 A_{Di}$. This leads to the same hamiltonian (3.21) which was obtained using the action (3.8). Moreover, there are the same constraints (3.20).

We conclude that at the first order in $\tilde\theta$ whatever the method used we obtain the same hamiltonian (3.21) and the constraints (3.20). However, the method of obtaining hamiltonian from the shifted action (3.8) seems easier: When the higher orders in $\tilde\theta$ are considered the unique change will be in the constraint (3.18), the other constraints (3.13)–(3.15), (3.17) will remain intact. Thus, finding hamiltonian of the dual theory is reduced to find solution of a constraint.

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4. BPS States of non-commutative D3-brane

In the zero slope limit, $\alpha' \to 0$, and considering slowly varying fields, noncommutative DBI action can be approximated as noncommutative gauge theory (3.21), up to constant terms [1]. Noncommutative D3-brane worldvolume action can be extracted from 10 dimensional noncommutative gauge theory in the static gauge. The first three spatial coordinates are taken equal to brane worldvolume coordinates and the rest of the coordinates are regarded as scalar fields on the brane. We consider only one scalar field. D3-brane worldvolume hamiltonian density resulting from (3.21) when $\tilde{\theta}^{0i} \neq 0$, $\tilde{\theta}^{ij} = 0$, is

$$\begin{aligned} H = & \frac{1}{2} {P_i}^2 + \frac{1}{4} F_{ij}^2 - \frac{1}{2} \tilde{\theta}^{0i} P_i P_j^2 - \frac{1}{4} \tilde{\theta}^{0i} P_i F_{jk}^2 + \tilde{\theta}^{0i} F_{ij} F_{jk} P^k + \\ & + \frac{1}{2} \pi^2 + \frac{1}{2} (\partial_i \phi)^2 - \frac{1}{2} \tilde{\theta}^{0i} P_i \pi^2 + \tilde{\theta}^{0i} \pi F_{ij} \partial_j \phi - \\ & - \tilde{\theta}^{0i} P_j \partial_i \phi \partial_j \phi + \frac{1}{2} \tilde{\theta}^{0i} P_i (\partial_j \phi)^2 . \end{aligned} \tag{4.1}$$

The scalar field and the corresponding canonical momentum denoted as ϕ and π. Moreover, we renamed the dual variables F_D, P_D as F, P. We choose $\pi = 0$ to deal with the static case.

To discuss bounds on the value of the hamiltonian we would like to write (4.1) as

$$H = \frac{1}{2} \widehat{P}_i^2 + \frac{1}{2} \widehat{B}_i^2 + \frac{1}{2} (\widehat{\partial_i \phi})^2 , \tag{4.2}$$

with the restrictions

$$\widehat{P}_i|_{P=0} = 0 , \qquad \widehat{B}_i|_{F=0} = 0 , \qquad \widehat{\partial_i \phi}|_{\phi=0} = 0 .$$

These are fulfilled by

$$\widehat{P}_i = P_i - a_1 \tilde{\theta}^{oi} P_j^2 - a_2 \tilde{\theta}^{oj} P_j P_i , \tag{4.3}$$

$$\widehat{B}_i = \frac{1}{2} \epsilon_{ijk} \left(F_{jk} - \frac{1}{2} \tilde{\theta}^{0l} P_l F_{jk} + b_1 \tilde{\theta}^{0l} P_k F_{jl} + b_2 \tilde{\theta}^{0k} P_l F_{jl} \right) , \tag{4.4}$$

$$\widehat{\partial_i \phi} = \partial_i \phi + \frac{1}{2} \tilde{\theta}^{0j} P_j \partial_i \phi - c_1 \tilde{\theta}^{0i} \partial_j \phi P_j - c_2 \tilde{\theta}^{0j} \partial_j \phi P_i , \tag{4.5}$$

where $a_{1,2}$ $b_{1,2}$ $c_{1,2}$ are constants which should satisfy

$$a_1 + a_2 = \frac{1}{2} , \qquad b_1 + b_2 = -2 , \qquad c_1 + c_2 = 1 , \tag{4.6}$$

otherwise arbitrary. These do not correspond to the Seiberg-Witten map (3.6). There the fields of commutative and noncommutative gauge theories are mapped into each other by changing the gauge group from the ordinary U(1) to noncommutative one such that (3.5) is satisfied. In our case gauge group is always U(1). Although we write the hamiltonian (4.2) in terms of $\tilde{\theta}^{0i}$ dependent (noncommutative) fields still there is the constraint

$$\partial_i P_i = 0 , \tag{4.7}$$

indicating U(1) gauge group. Seiberg-Witten map in phase space is studied in [9]–[13].

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Now, in terms of an arbitrary angle α the hamiltonian density (4.1) can be put into the form

$$H = \frac{1}{2}\left(\widehat{P}_i - \sin\alpha\ \widehat{\partial_i\phi}\right)^2 + \frac{1}{2}\left(\widehat{B}_i - \cos\alpha\ \widehat{\partial_i\phi}\right)^2 + \\ + \sin\alpha\ \widehat{P}_i\widehat{\partial_i\phi} + \cos\alpha\ \widehat{B}_i\widehat{\partial_i\phi}\,. \tag{4.8}$$

Thus, we can write a bound on the total energy E relative to the worldvolume vacuum of noncomutative D3-brane as

$$E \geq \sqrt{\tilde{Z}_{\rm el}^2 + \tilde{Z}_{\rm mag}^2}\,, \tag{4.9}$$

where, we introduced

$$\tilde{Z}_{\rm el} = \int_{D3} d^3x\ \widehat{P}_i\widehat{\partial_i\phi}\,, \tag{4.10}$$

$$\tilde{Z}_{\rm mag} = \int_{D3} d^3x\ \widehat{B}_i\widehat{\partial_i\phi}\,. \tag{4.11}$$

In the commutative case $\tilde{Z}_{\rm el}$ and $\tilde{Z}_{\rm mag}$ become topological charges due to the Gauss law and the Bianchi identity: $\partial_i P_i = 0$, $\partial_i B_i = 0$, moreover, (4.9) is known as BPS bound [14, 15]. However, in our case we do not have integrability conditions for $\widehat{P}_i$, $\widehat{B}_i$. Nevertheless, it will be shown that $\tilde{Z}_{\rm el}$, $\tilde{Z}_{\rm mag}$ can be topological charges for some specific configurations:

The bound (4.9) is saturated for

$$\widehat{P}_i = \widehat{\partial_i\phi}\,, \qquad \widehat{B}_i = 0\,, \qquad \sin\alpha = 1\,. \tag{4.12}$$

This can be accomplished at the first order in $\tilde{\theta}^{0i}$, when

$$F_{ij} = 0\,, \qquad P_i = \partial_i\phi\,, \tag{4.13}$$

if we fix the parameters as

$$a_1 = c_1\,, \qquad a_2 = c_2 - \frac{1}{2}\,, \tag{4.14}$$

which are consistent with (4.6). Because of the constraint (4.7), ϕ should satisfy

$$\partial_i^2\phi = 0\,. \tag{4.15}$$

For this configuration $\tilde{Z}_{\rm mag}$ vanishes: $\tilde{Z}_{\rm mag}^{(1)} = 0$, and $\tilde{Z}_{\rm el}$ reads

$$\tilde{Z}_{\rm el}^{(1)} = \int_{D3} d^3x\partial_i(\phi\partial_i\phi) - \int_{D3} d^3x\ \tilde{\theta}^{0i}\partial_i\phi(\partial\phi)^2\,. \tag{4.16}$$

For the commutative case isolated singularities of ϕ satisfying these conditions are called BIon [14]. The simplest choice satisfying (4.15) is [15]

$$\phi(r) = \frac{e}{4\pi r}\,, \tag{4.17}$$

where r is the radial variable. In general we cannot write $\tilde{\theta}^{0i}$ dependent part as a surface integral. However, this choice of harmonic function (4.17) renders it possible. Indeed, we can write $\tilde{Z}_{\rm el}^{(1)}$ as an integral over a sphere of radius ϵ about the origin and find

$$\tilde{Z}_{\rm el}^{(1)} = \left(e - \frac{\tilde{\theta}e^2}{20\pi\epsilon^4}\right)\lim_{\epsilon\to 0}\phi(\epsilon)\,, \tag{4.18}$$

where $\tilde{\theta} \equiv \sqrt{\tilde{\theta}^{0i}\tilde{\theta}^{0i}}$.

Observe that the usual BIon solution (4.17) leads to a solution for the noncommutative case (4.12). This is similar to the fact that linearized and full DBI actions lead to the same BIon solution with the same energy [6]. Here solutions are the same but energies differ.

When one sets $P_i = 0$ the terms depending on the noncommutativity parameter $\tilde{\theta}^{0i}$ disappear. This is what we expected: Noncommutativity is only between time and space coordinates, not between spatial coordinates. Thus, when momenta vanish noncommutativity should cease to exist. For $P_i = 0$, the bound (4.9) is saturated for

$$\frac{1}{2}\epsilon_{ijk}F_{jk} = \partial_i\phi\,, \qquad \cos\alpha = 1 \tag{4.19}$$

where as before ϕ should satisfy (4.15). For this commuting configuration $\tilde{Z}_{\rm el}$ and $\tilde{Z}_{\rm mag}$ are given as $\tilde{Z}_{\rm el}^{(2)} = 0$, and

$$\tilde{Z}_{\rm mag}^{(2)} = \int_{D3} d^3x\partial_i(\phi\partial_i\phi)\,. \tag{4.20}$$

To satisfy (4.19) and (4.15) consider a magnetic charge at the origin

$$\phi(r) = \frac{g}{4\pi r}\,. \tag{4.21}$$

Let the integral be over a sphere of radius ϵ about the origin which yields

$$\tilde{Z}_{\rm mag}^{(2)} = g\lim_{\epsilon\to 0}\phi(\epsilon)\,. \tag{4.22}$$

There is another configuration

$$\widehat{P}_i = \sin\alpha\widehat{\partial_i\phi}\,, \qquad \widehat{B}_i = \cos\alpha\widehat{\partial_i\phi}\,, \tag{4.23}$$

which saturates the bound (4.9). The constant angle α is defined as

$$\tan\alpha = \frac{\tilde{Z}_{\rm el}}{\tilde{Z}_{\rm mag}}\,.$$

This can be realized if the commuting variables are fixed as

$$P_i = \sin\alpha\ \partial_i\phi\,, \qquad \frac{1}{2}\epsilon_{ijk}F_{jk} = \cos\alpha\ \partial_i\phi \tag{4.24}$$

and the free parameters in (4.3)–(4.5) satisfy (4.14) and

$$c_1 = \frac{b_1}{2}\,, \qquad c_2 = 1 - \frac{b_1}{2}\,. \tag{4.25}$$

These are consistent with (4.6). Thus, in the hatted quantities (4.3)–(4.5) now, there is only one free constant parameter. For this configuration $\tilde{Z}_{\rm el}$ and $\tilde{Z}_{\rm mag}$ are given by

$$\tilde{Z}_{\rm el}^{(3)} = \int_{D3} d^3x\sin\alpha\ \partial_i(\phi\partial_i\phi) - \int_{D3} d^3x\tilde{\theta}^{0i}\sin^2\alpha\ \partial_i\phi(\partial\phi)^2\,, \tag{4.26}$$

$$\tilde{Z}_{\rm mag}^{(3)} = \int_{D3} d^3x\cos\alpha\ \partial_i(\phi\partial_i\phi) - \int_{D3} d^3x\tilde{\theta}^{0i}\cos^2\alpha\ \partial_i\phi(\partial\phi)^2\,. \tag{4.27}$$

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Similar to the other configurations, ϕ should satisfy (4.15) and we consider the simplest choice

$$\phi(r) = \frac{g}{4\pi \cos\alpha \; r} \,. \tag{4.28}$$

For this choice of the harmonic function (4.28) the integrals in (4.26) and (4.27) can be performed over a sphere of radius ϵ about the origin. Therefore, the energy can be calculated as

$$E = \left[\left(e - \frac{\tilde{\theta} e^2}{20\pi\epsilon^4} \right)^2 + \left(g - \frac{\tilde{\theta} g^2}{20\pi\epsilon^4} \right)^2 \right]^{1/2} \lim_{\epsilon \to 0} \phi(\epsilon) \,, \tag{4.29}$$

where $e/g = \tan\alpha$. Similar to the above mentioned configurations ordinary D3-brane dyon solution (4.28), provide a solution of the noncommutative condition (4.23).

5. Discussions

The results which we obtained are valid at the first order in the noncommutativity parameter θ. In principle contributions at higher orders in θ can be calculated. Obviously, one of the methods is to solve $\partial_0 A_D$ in terms of P_D, F_D from the generalization of (3.23). However, it is highly non-linear. On the other hand using the shifted action as it is illustrated here seems more manageable. We are encouraged from the fact that one should only solve a constraint similar to (3.18). The other constraints (3.13)–(3.15), (3.17) remain intact.

Noncommuting D3-brane formulation which we deal with is somehow different from the one considered in [2], [16]–[18]. There, gauge group is noncommutative U(1), in our case although hamiltonian depends on the noncommutativity parameter θ, gauge group is still U(1). This seems to be the basic reason that the BPS solutions of ordinary case [14]–[15] provide solutions of the noncommutative case as it happens between linearized and full DBI action [6].

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